A PARAMETRIC METHOD OF SOLVING LAMINAR BOUNDARY LAYER EQUATIONS WITH A LONGITUDINAL PRESSURE GRADIENT IN AN EQUILIBRIUM-DISSOCIATED GAS

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The possibility is examined of reducing the equations of a laminar boundary layer with longitudinal pressure gradient in the presence of an equilibrium-dissociated gas to some "universal" system of equations.

Basic system of equations. Transformation of variables. We shall examine the laminar boundary layer on a body of arbitrary shape in a high-speed gas stream. Halting the flow in a viscous boundary layer causes a sharp temperature increase, leading to dissociation. We shall assume that the reaction rates of dissociation and recombination are so high that thermochemical equilibrium is established throughout the entire boundary layer. The gas in the external stream is considered to be cold and undissociated.

In this case the equations of a plane steady boundary layer may be written in the form

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)=0,$$

$$\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=\rho_e u_e\frac{du_e}{dx}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right),$$

$$\rho u\frac{\partial h}{\partial x}+\rho v\frac{\partial h}{\partial y}=$$

$$=-u\rho_e u_e\frac{du_e}{dx}+\mu\left(\frac{\partial u}{\partial y}\right)^2+\frac{\partial}{\partial y}\left[\frac{\mu}{\Pr}(1+l)\frac{\partial h}{\partial y}\right], \quad (1)$$

with boundary conditions

$$u=v=0,\ h=h_w \text{ when } y=0,$$
$$u\to u_e(x),\ h\to h_e(x) \text{ when } y\to\infty,$$
$$u=u_0(y),\ h=h_0(y) \text{ when } x=x_0, \quad (2)$$

where $u_0(y)$ and $h_0(y)$ are given distributions of velocity and enthalpy at some initial section of the boundary layer with abscissa x_0, and $l = l(p, h)$ is a function which may be written, for a binary mixture, in the form

$$l=(\mathrm{Le}-1)(h_A-h_M)\left(\frac{\partial c_A}{\partial h}\right)_p. \quad (3)$$

Applying the Dorodnitsyn transformation, modified according to Lees for the longitudinal coordinate, to the system (1),

$$s(x)=\frac{1}{\rho_0\mu_0}\int_0^x\rho_w\mu_w\,dx, \qquad \eta(x,y)=\frac{1}{\rho_0}\int_0^y\rho\,dy,$$

where ρ_0, μ_0 are arbitrary constant values of density and dynamic viscosity, and introducing a stream function ψ according to the relation

$$u=\frac{\partial\psi}{\partial\eta},\quad \tilde{v}=\frac{\rho_0\mu_0}{\rho_w\mu_w}\left[u\frac{\partial\eta}{\partial x}+v\frac{\rho}{\rho_0}\right]=-\frac{\partial\psi}{\partial s},$$

we write the system (1), (2) in the form

$$\frac{\partial\psi}{\partial\eta}\frac{\partial^2\psi}{\partial s\,\partial\eta}-\frac{\partial\psi}{\partial s}\frac{\partial^2\psi}{\partial\eta^2}=\frac{\rho_e}{\rho}u_e\frac{du_e}{ds}+\nu_0\frac{\partial}{\partial\eta}\left(N\frac{\partial^2\psi}{\partial\eta^2}\right),$$

$$\frac{\partial\psi}{\partial\eta}\frac{\partial h}{\partial s}-\frac{\partial\psi}{\partial s}\frac{\partial h}{\partial\eta}=-\frac{\rho_e}{\rho}u_e\frac{du_e}{ds}\frac{\partial\psi}{\partial\eta}+\nu_0N\left(\frac{\partial^2\psi}{\partial\eta^2}\right)^2$$

$$+\nu_0\frac{\partial}{\partial\eta}\times\left[\frac{N}{\Pr}(1+l)\frac{\partial h}{\partial\eta}\right],$$

$$\psi=\frac{\partial\psi}{\partial\eta}=0,\ h=h_w \text{ when } \eta=0,$$

$$\frac{\partial\psi}{\partial\eta}\to u_e(s),\ h\to h_e(s) \text{ when } \eta\to\infty,$$

$$\frac{\partial\psi}{\partial\eta}=u_0(\eta),\ h=h_0(\eta) \text{ when } s=s_0, \quad (4)$$

where $\nu_0 = \mu_0/\rho_0$ is the dynamic viscosity.

The function N is determined as follows:

$$N=\rho\mu/\rho_w\mu_w;\ N=1 \text{ when } \eta=0,$$
$$N\to\rho_e\mu_e/\rho_w\mu_w=N(s) \text{ when } \eta\to\infty.$$

The momentum equation, which is easily obtained from (4), may be written in one of the following forms, no different from that of the momentum equation for an incompressible fluid (here and below the prime denotes differentiation with respect to s):

$$\frac{dz^{**}}{ds}=\frac{F}{u_e},\quad \frac{df}{ds}=\frac{u_e'}{u_e}F+\frac{u_e''}{u_e'}f,$$

$$\frac{1}{\Delta^{**}}\frac{d\Delta^{**}}{ds}=\frac{F}{2f}\frac{u_e'}{u_e}, \quad (5)$$

if the conventional boundary layer thicknesses Δ^* (displacement thickness) and Δ^{**} (momentum thickness) are introduced in the form

$$\Delta^*=\int_0^\infty\left(\frac{\rho_e}{\rho}-\frac{u}{u_e}\right)d\eta,\quad \Delta^{**}=\int_0^\infty\frac{u}{u_e}\left(1-\frac{u}{u_e}\right)d\eta. \quad (6)$$

It is assumed in (5) and (6) that

$$z^{**}=\frac{\Delta^{**2}}{\nu_0},\quad \zeta=\left[\frac{\partial(u/u_e)}{\partial(\eta/\Delta^{**})}\right]_w,$$

$$H=\frac{\Delta^*}{\Delta^{**}},\quad F=2[\zeta-(2+H)f],\quad f=\frac{u_e'(\Delta^{**})^2}{\nu_0}. \quad (7)$$

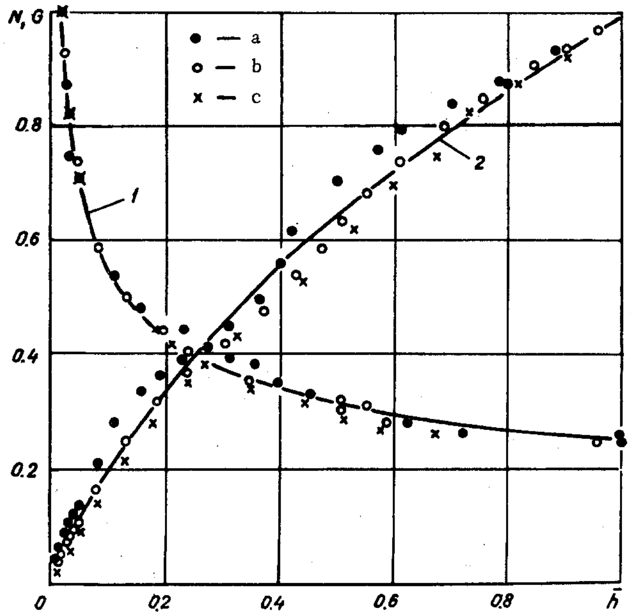

Fig. 1. Dependence of $N = \rho\mu/\rho_w\mu_w$ (1) and density ratio $G = \rho_1/\rho$ (2) on the dimensionless enthalpy $\bar{h}$ at $M_\infty = 19.4$, $h_w = 0.0152$: 1), 2) according to (13), (14); a), b), c) from tables of thermodynamic functions of air, with p equal to 10^{-4} p_a, 0.35 p_a, and 10 p_a respectively.

Following [1], we shall go over in (4) to the new variables

$$s = s,\quad \xi = B\eta/\Delta^{**},\quad \Phi = B\psi/u_e\Delta^{**},\quad \bar{h} = h/h_1, \tag{8}$$

where B is some normalizing multiplier.

Using (8), we obtain the following system of two differential equations for the "reduced" stream function Φ and the dimensionless enthalpy $\bar{h}$:

$$\frac{\partial}{\partial\xi}\left(N\frac{\partial^2\Phi}{\partial\xi^2}\right) + \frac{F+2f}{2B^2}\Phi\frac{\partial^2\Phi}{\partial\xi^2} + \frac{f}{B^2}\left[\frac{\rho_e}{\rho} - \left(\frac{\partial\Phi}{\partial\xi}\right)^2\right] = \frac{u_e f}{u_e' B}\left(\frac{\partial\Phi}{\partial\xi}\frac{\partial^2\Phi}{\partial s\partial\xi} - \frac{\partial\Phi}{\partial s}\frac{\partial^2\Phi}{\partial\xi^2}\right),$$

$$\frac{\partial}{\partial\xi}\left[\frac{N(1+l)}{\Pr}\frac{\partial\bar{h}}{\partial\xi}\right] + \frac{F+2f}{2B^2}\Phi\frac{\partial\bar{h}}{\partial\xi} - \frac{2\varkappa f}{B^2}\frac{\rho_e}{\rho}\frac{\partial\Phi}{\partial\xi} + 2\varkappa N\left(\frac{\partial^2\Phi}{\partial\xi^2}\right)^2 = \frac{u_e f}{u_e' B}\left(\frac{\partial\Phi}{\partial\xi}\frac{\partial\bar{h}}{\partial s} - \frac{\partial\Phi}{\partial s}\frac{\partial\bar{h}}{\partial\xi}\right). \tag{9}$$

$$\Phi = \frac{\partial\Phi}{\partial\xi} = 0,\quad \bar{h} = \bar{h}_w \text{ when } \xi = 0,$$

$$\frac{\partial\Phi}{\partial\xi} \to 1,\quad \bar{h} \to 1 - \varkappa(s) \text{ when } \xi \to \infty,$$

$$\Phi = \Phi_0(\xi),\quad \bar{h} = \bar{h}_0(\xi) \text{ when } s = s_0.$$

Equation (9) contains the quantity $\varkappa = u_e^2/2h_1$, given as a function of s, which may be called the "local compressibility factor" of the gas and may be expressed in terms of the Mach number $M_e = u_e/a_e$ of the flow outside the boundary layer.

Any self-similar solutions of system (9) may be chosen as functions $\Phi_0(\xi)$ and $\bar{h}_0(\xi)$. We shall consider later that $\Phi_0(\xi)$ and $\bar{h}_0(\xi)$ are solutions of the system of equations describing the flow in the laminar boundary layer on a flat plate (u_e = const, $f = 0$, $\varkappa = \varkappa_0 = u_\infty^2/2h_1$):

$$\frac{d}{d\xi}\left[N\frac{d^2\Phi_0}{d\xi^2}\right] + \Phi_0\frac{d^2\Phi_0}{d\xi^2} = 0,$$

$$\frac{d}{d\xi}\left[\frac{N}{\Pr}\frac{d\bar{h}_0}{d\xi}\right] + \Phi_0\frac{d\bar{h}_0}{d\xi} + 2\varkappa_0 N\left(\frac{d^2\Phi_0}{d\xi^2}\right)^2 = 0,$$

$$\Phi_0 = \frac{d\Phi_0}{d\xi} = 0,\quad \bar{h}_0 = \bar{h}_w \text{ when } \xi = 0,$$

$$\frac{d\Phi_0}{d\xi} \to 1,\quad \bar{h}_0 \to 1 - \varkappa_0 \text{ when } \xi \to \infty. \tag{10}$$

The normalizing multiplier B is chosen such that for a constant velocity at the edge of the boundary layer $u_e = u_\infty$, system (9) converts to system (10); then

$$B = \int_0^\infty \frac{d\Phi_0}{d\xi}\left(1 - \frac{d\Phi_0}{d\xi}\right)d\xi = \sqrt{\frac{F_0}{2}}.$$

We shall determine the functions N, ρ_e/ρ, l, Pr which depend on the thermodynamic and transport properties of an equilibrium-dissociated gas.

Assumptions and approximation formula. The Pr and Le numbers depend weakly on temperature up to temperatures of the order of $T \cong 9000°$ K [2], and we shall therefore consider them to be constant; in addition, we shall take the Lewis number to be unity (Le = 1). Calculation shows that this approximation is quite acceptable for equilibrium dissociation [2]. With this assumption, from (3), $l = 0$. The Prandtl number is assumed to be 0.712 in the calculations that follow.

Main Characteristics of the Boundary Layer as a Function of the "Local Compressibility Factor" with $f_1 = 0$

Characteristics of boundary layer	Numerical values of quantities with $\varkappa$ equal to:									
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
ζ_0	0.0749	0.0762	0.0777	0.0793	0.0811	0.0832	0.0855	0.0882	0.0916	0.0960
ζ_0^*	0.0615	0.0618	0.0651	0.0654	0.0659	0.0665	0.0673	0.0683	0.0697	0.0717
F_0	0.1497	0.1524	0.1554	0.1586	0.1622	0.1663	0.1710	0.1765	0.1832	0.1920
H_0	0.3655	0.4562	0.5662	0.6958	0.8548	1.0651	1.3801	1.9296	3.1035	6.8478

The quantities $N = \rho\mu/\rho_w\mu_w$ and ρ_e/ρ are, in general, functions of the dimensionless enthalpy and the pressure.

For large M_∞ in a dissociated gas the quantity $\rho\mu$ changes strongly through the boundary layer, and N cannot be regarded as constant.

The viscosity of air in equilibrium is well described up to about T = 4000° K by the Sutherland formula

$$\mu/\mu_w = (T/T_w)^{3/2}(T_w + 112°K)/(T + 112°K). \qquad (11)$$

At T ≥ 4000° K Eq. (11) will give values that are somewhat low. An appropriate correction may be applied, for instance, on the basis of Hansen's paper [3].

In the density ratio appearing in (9) we separate out the factor ρ_e/ρ_1, which depends only on conditions in the external flow:

$$\frac{\rho_e}{\rho} = \frac{\rho_e}{\rho_1}\frac{\rho_1}{\rho}. \qquad (12)$$

The density ratios ρ_1/ρ and ρ/ρ_w (when $T = T_w =$ = const) may be determined from tables of thermodynamic functions of air as a function of $\bar{h}$ and p. The following approximate formulas may be put forward for N and ρ_1/ρ over the wide pressure range $10^{-4} \le$ $\le p/p_a \le 10$ (where p_a is atmospheric pressure):

$$N = N(\bar{h}) = \sqrt[3]{\bar{h}_w/\bar{h}}, \qquad (13)$$

$$\rho_1/\rho = G(\bar{h}) = a_1\bar{h} + a_2\bar{h}^2 + a_3\bar{h}^3 + a_4\bar{h}^4, \qquad (14)$$

where the coefficients a_k depend on the Mach number of the undisturbed stream. It may be seen from Fig. 1 that these approximations provide satisfactory accuracy (the maximum error does not exceed 15%).

Using (12) and (14), the density ratio ρ_e/ρ in (9) may be written in the form

$$\rho_e/\rho = G(\bar{h})/G(1-\varkappa), \qquad (15)$$

where we have put

$$G(1-\varkappa) = a_1(1-\varkappa) + a_2(1-\varkappa)^2 + a_3(1-\varkappa)^3 + a_4(1-\varkappa)^4,$$
$$\varkappa = u_e^2/2h_1 = \varkappa(s).$$

Conversion of (9) to "universal" form. The solution of (9) will depend on the specific form in which the velocity at the edge of the boundary layer is assigned, as well as on a number of constant parameters (M_∞, Pr, $\bar{h}_w$). The system (9) may be made independent of the velocity distribution at the outer edge of the boundary layer, using the method suggested by Loitsyanskii [1] of making the boundary layer equations universal, on the basis of conversion of the parameters expressing the external flow conditions into a number of independent variables.

Let us introduce the infinite system of parameters

$$f_0 = u_e^2/2h_1, \quad f_k = u_e^{k-1}u_e^{(k)}z^{**k}\,(k = 1, 2\ldots),$$

where $f_0 = \varkappa$ is the local compressibility factor, and f_k (k = 1, 2...) are parameters expressing the shape of the body; when k = 1 we have $f_1 = u_e'\Delta^{**2}/\nu_0$, the known shape factor of boundary layer theory. Carrying out the substitution of variables in system (9)

$$\frac{\partial}{\partial s} = \sum_{k=1}^{\infty}\frac{df_k}{ds}\frac{\partial}{\partial f_k} + \frac{d\varkappa}{ds}\frac{\partial}{\partial\varkappa},$$

and taking into account that

$$\frac{u}{u_e'}\frac{d\varkappa}{ds} = 2\varkappa = \Theta_0,$$

$$\frac{u_e}{u_e'}f_1\frac{df_1}{ds} = [(k-1)f_1 + kF]f_k + f_{k+1} = \Theta_k\,(\text{when } k \geqslant 1),$$

we arrive at the following "universal" system of equations

$$\frac{\partial}{\partial\xi}\left[N(\bar{h})\frac{\partial^2\Phi}{\partial\xi^2}\right] + \frac{F+2f_1}{2B^2}\Phi\frac{\partial^2\Phi}{\partial\xi^2} + \frac{f_1}{B^2}\left[\frac{G(\bar{h})}{G(1-\varkappa)} - \left(\frac{\partial\Phi}{\partial\xi}\right)^2\right] =$$
$$= \frac{1}{B^2}\sum_{k=0}^{\infty}\Theta_k\left(\frac{\partial\Phi}{\partial\xi}\frac{\partial^2\Phi}{\partial\xi\partial f_k} - \frac{\partial\Phi}{\partial f_k}\frac{\partial^2\Phi}{\partial\xi^2}\right),$$

$$\frac{\partial}{\partial\xi}\left[\frac{N(\bar{h})}{\Pr}\frac{\partial\bar{h}}{\partial\xi}\right] + \frac{F+2f_1}{2B^2}\Phi\frac{\partial\bar{h}}{\partial\xi} - 2\frac{\varkappa f_1}{B^2}\frac{G(\bar{h})}{G(1-\varkappa)}\frac{\partial\Phi}{\partial\xi} +$$
$$+ 2\varkappa N(\bar{h})\left(\frac{\partial^2\Phi}{\partial\xi^2}\right)^2 = \frac{1}{B^2}\sum_{k=0}^{\infty}\Theta_k\left(\frac{\partial\Phi}{\partial\xi}\frac{\partial\bar{h}}{\partial f_k} - \frac{\partial\Phi}{\partial f_k}\frac{\partial\bar{h}}{\partial\xi}\right).$$

$$\Phi = \frac{\partial\Phi}{\partial\xi} = 0,\ \bar{h} = \bar{h}_w = \text{const when } \xi = 0,$$
$$\frac{\partial\Phi}{\partial\xi} \to 1,\ \bar{h} \to 1-\varkappa \text{ when } \xi \to \infty,$$
$$\Phi = \Phi_0(\xi),$$
$$\bar{h} = \bar{h}_0(\xi) \text{ when } f_0 = \varkappa_0 = u_\infty^2/2h_1;\ f_1 = f_2 = \ldots = 0, \qquad (16)$$

where $\Phi_0(\xi)$ and $\bar{h}_0(\xi)$ are solutions of the self-similar system (10), and the functions G(h), G(1 − ϰ), N($\bar{h}$) are determined according to formulas (11)–(15).

The final solution of each specific problem with its given velocity distribution $u_e = u_e(s)$ at the outer edge of the boundary layer requires integration of the ordinary differential equation of first order

$$\frac{dz^{**}}{ds} = \frac{F(f_0, f_1, f_2\ldots)}{u_e(s)} = \frac{F\left(\frac{u_e^2}{2h_1};\ u_e'z^{**};\ u_eu_e''z^{**2};\ \ldots\right)}{u_e(s)},$$

where, according to (6) and (7),

$$F = 2[\zeta - (2+H)f_1],\quad H = \frac{1}{B}\int_0^\infty\left[\frac{G(\bar{h})}{G(1-\varkappa)} - \frac{\partial\Phi}{\partial\xi}\right]d\xi,$$

$$\zeta = B\left(\frac{\partial^2 \Phi}{\partial \xi^2}\right)_{\xi=0}, \quad f_0 = \frac{u_e^2}{2h_1} = \varkappa, \; f_1 = u_e' z^{**} = \frac{\Delta^{**^2}}{\nu_0}\; u_e'.$$

Approximate solutions of the "universal" system of equations. Numerical solution of the "universal" system of equations for a large number of parameters presents insurmountable computational difficulties. In solving (16) we must be content with the minimum possible number of independent variables.

Functions Φ and $\bar{h}$ of (16) depend on a combination of the arguments ξ, $f_0 = \varkappa = u_e^2/2h_1$, $f_1 = u_e' z^{**}$. The first in importance after ξ in the series of arguments is the local compressibility factor $\varkappa$, then comes the system of boundary layer shape factors determined by successive derivatives of the velocity at the edge of the boundary layer.

The method of successive approximations may be suggested as a method of solving (16). In the first approximation we put all the $f_k = 0$. Then all the derivatives with respect to $\varkappa$ automatically vanish, and $\varkappa$ may be regarded as a parameter. In this approximation the system of equations (16) reduces to (10), and the parameter $\varkappa$ should be given a discrete number of values in the range $(0 \le \varkappa < 1)$.

The system (10) was integrated by the Runge-Kutta method on an electronic computer with the following values of the constant parameters: Pr = 0.712, $\bar{h}_w$ = = 0.0152 (cooled wall). Tabulated values of the functions $\Phi_0(\xi)$ and $\bar{h}_0(\xi)$ and their derivatives resulted from the solution. The table presents values of the reduced friction coefficient at the wall $\zeta_0 = B\left(\frac{\partial^2 \Phi_0}{\partial \xi^2}\right)_{\xi=0}$ and the reduced heat flux to the wall $\zeta_0^* = B\left(\frac{\partial \bar{h}_0}{\partial \xi}\right)_{\xi=0}$, as well as the quantities F_0 and H_0 as a function of parameter $\varkappa$.

It may be seen from an examination of the table and the graphs of Figs. 2a and b that the influence of the parameter $\varkappa$ on the quantities ζ_0, ζ_0^* and on the velocity profiles in the boundary layer is negligibly small.

The quantities H_0 and the enthalpy distribution across the boundary layer, on the other hand, vary sharply with change of $\varkappa$, and even the general behavior of the enthalpy in the boundary layer changes. When $\varkappa \leqslant 0.6$ the enthalpy $\bar{h}$ reaches its maximum value, equal to $1 - \varkappa$, at the outer edge at the boundary layer. When $\varkappa > 0.6$ the enthalpy has a maximum $(\bar{h}_{max} > 1 - \varkappa)$ inside the boundary layer.

In the next approximation, putting all the $f_k = 0$ with $k \geq 2$ in (16), we obtain a system of equations in partial derivatives with respect to the three independent variables ξ, $\varkappa$, f_1:

$$\frac{\partial}{\partial \xi}\left[N(\bar{h})\frac{\partial^2 \Phi}{\partial \xi^2}\right] + \frac{F + 2f_1}{2B^2}\Phi\frac{\partial^2 \Phi}{\partial \xi^2} + \frac{f_1}{B^2}\left[\frac{G(\bar{h})}{G(1-\varkappa)} - \left(\frac{\partial \Phi}{\partial \xi}\right)^2\right] = \frac{2\varkappa f_1}{B^2}\left(\frac{\partial \Phi}{\partial \xi}\frac{\partial^2 \Phi}{\partial \xi \partial \varkappa} - \frac{\partial \Phi}{\partial \varkappa}\frac{\partial^2 \Phi}{\partial \xi^2}\right) + \frac{F f_1}{B^2}\left(\frac{\partial \Phi}{\partial \xi}\frac{\partial^2 \Phi}{\partial \xi \partial f_1} - \frac{\partial \Phi}{\partial f_1}\frac{\partial^2 \Phi}{\partial \xi^2}\right),$$

$$\frac{\partial}{\partial \xi}\left[\frac{N(\bar{h})}{\Pr}\frac{\partial \bar{h}}{\partial \xi}\right] + \frac{F + 2f_1}{2B^2}\Phi\frac{\partial \bar{h}}{\partial \xi} - 2\frac{\varkappa f_1}{B^2}\frac{G(\bar{h})}{G(1-\varkappa)}\frac{\partial \Phi}{\partial \xi} + 2\varkappa N(\bar{h})\left(\frac{\partial^2 \Phi}{\partial \xi^2}\right)^2 = \frac{2\varkappa f_1}{B^2}\left(\frac{\partial \Phi}{\partial \xi}\frac{\partial \bar{h}}{\partial \varkappa} - \frac{\partial \Phi}{\partial \varkappa}\frac{\partial \bar{h}}{\partial \xi}\right) + \frac{F f_1}{B^2}\left(\frac{\partial \Phi}{\partial \xi}\frac{\partial \bar{h}}{\partial f_1} - \frac{\partial \Phi}{\partial f_1}\frac{\partial \bar{h}}{\partial \xi}\right),$$

$$\Phi = \frac{\partial \Phi}{\partial \xi} = 0, \; \bar{h} = \bar{h}_w = \text{const when } \xi = 0,$$

$$\frac{\partial \Phi}{\partial \xi} \to 1, \; \bar{h} \to (1-\varkappa) \text{ when } \xi \to \infty,$$

$$\Phi = \Phi_0(\xi), \quad \bar{h} = \bar{h}_0(\xi) \text{ when } \varkappa = \varkappa_0; \; f_1 = 0,$$

where $\Phi_0(\xi)$, $\bar{h}_0(\xi)$ are solutions of the system of equations (10), and $\varkappa_0$ is determined by assigning the number M_∞.

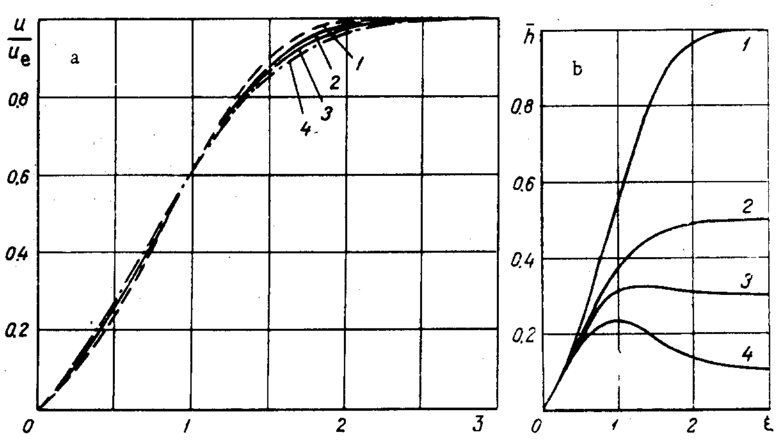

Fig. 2. Profiles of velocity $u/u_e = \partial\Phi_0/\partial\xi$ (a) and enthalpy $\bar{h}$ (b) over the section of the boundary layer when $f_1 = 0$ and with values of the parameter $\varkappa$. 1) 0; 2) 0.5; 3) 0.7; 4) 0.9.

Because of the complexity of direct numerical integration of system (17) with three independent variables, we shall discuss the possibility of an approximate solution of (17).

Let us introduce the dimensionless total enthalpy

$$g = \frac{h + u^2/2}{h_1},$$

and, expressing $\bar{h}$ in the second equation of system (17) in terms of g according to the relation

$$\bar{h} = g - \varkappa \left(\frac{\partial \Phi}{\partial \xi} \right)^2,$$

represent the energy equation in the form

$$\frac{\partial}{\partial \xi} \left[\frac{N(\bar{h})}{\Pr} \frac{\partial g}{\partial \xi} \right] + \frac{F + 2f_1}{2B^2} \Phi \frac{\partial g}{\partial \xi} + 2\varkappa \frac{\Pr - 1}{\Pr} \frac{\partial}{\partial \xi} \left[\frac{\partial \Phi}{\partial \xi} N(\bar{h}) \frac{\partial^2 \Phi}{\partial \xi^2} \right] = \frac{2\varkappa f_1}{B^2} \left(\frac{\partial \Phi}{\partial \xi} \frac{\partial g}{\partial \varkappa} - \frac{\partial \Phi}{\partial \varkappa} \frac{\partial g}{\partial \xi} \right) + \frac{F f_1}{B^2} \left(\frac{\partial \Phi}{\partial \xi} \frac{\partial g}{\partial f_1} - \frac{\partial \Phi}{\partial f_1} \frac{\partial g}{\partial \xi} \right),$$

$$g = g_w \text{ when } \xi = 0, \quad g \to 1 \text{ when } \xi \to \infty,$$

$$g = g_0(\xi) \text{ when } \varkappa = \varkappa_0, \; f_1 = 0. \tag{18}$$

It may be seen from an examination of the solution of the "universal system" (16) in the first approximation ($f_k = 0$ when $k \geq 1$) that the influence of the compressibility factor $\varkappa = \varkappa(M_e)$ on the reduced stream function is inappreciable. This is evidently connected with the fact that the effect of M_∞ is partially allowed for by the Dorodnitsyn transformation. It is apparent that this influence will also be weak in relation to the total enthalpy. If we suppose that this evaluation also obtains when $f_1 \neq 0$, then to obtain an approximate solution we may neglect the derivative $\partial\Phi/\partial\varkappa$, $\partial^2\Phi/\partial\xi\partial\varkappa$, $\partial g/\partial\varkappa$ in (17) and (18).

It should be noted that the derivative $\partial\bar{h}/\partial\varkappa$ must not be put equal to zero, since the influence of the parameter $\varkappa$ on the dimensionless enthalpy is considerable.

Putting $\partial\Phi/\partial\varkappa = 0$, $\partial^2\Phi/\partial\xi\partial\varkappa = 0$, $\partial g/\partial\varkappa = 0$ in (17) and (18), we then return to the enthalpy $\bar{h}$, since the quantities $N = \rho\mu/\rho_W\mu_W$ and $G = \rho_1/\rho$ appearing in the equations of the system are functions (13) and (14) of $\bar{h}$. We finally have the following approximate system:

$$\frac{\partial}{\partial \xi} \left[N(\bar{h}) \frac{\partial^2 \Phi}{\partial \xi^2} \right] + \frac{F + 2f_1}{2B^2} \Phi \frac{\partial^2 \Phi}{\partial \xi^2} + \frac{f_1}{B^2} \left[\frac{G(\bar{h})}{G(1-\varkappa)} - \left(\frac{\partial \Phi}{\partial \xi} \right)^2 \right] = \frac{F f_1}{B^2} \left(\frac{\partial \Phi}{\partial \xi} \frac{\partial^2 \Phi}{\partial \xi \partial f_1} - \frac{\partial \Phi}{\partial f_1} \frac{\partial^2 \Phi}{\partial \xi^2} \right),$$

$$\frac{\partial}{\partial \xi} \left[\frac{N(\bar{h})}{\Pr} \frac{\partial \bar{h}}{\partial \xi} \right] + \frac{F + 2f_1}{2B^2} \Phi \frac{\partial \bar{h}}{\partial \xi} - 2 \frac{\varkappa f_1}{B^2} \frac{\partial \Phi}{\partial \xi} \left[\frac{G(\bar{h})}{G(1-\varkappa)} - \left(\frac{\partial \Phi}{\partial \xi} \right)^2 \right] + 2\varkappa N(\bar{h}) \times \left(\frac{\partial^2 \Phi}{\partial \xi^2} \right)^2 = \frac{F f_1}{B^2} \left(\frac{\partial \Phi}{\partial \xi} \frac{\partial \bar{h}}{\partial f_1} - \frac{\partial \Phi}{\partial f_1} \frac{\partial \bar{h}}{\partial \xi} \right),$$

$$\Phi = \frac{\partial \Phi}{\partial \xi} = 0, \quad \bar{h} = \bar{h}_w \text{ when } \xi = 0,$$

$$\frac{\partial \Phi}{\partial \xi} \to 1, \quad \bar{h} \to 1 - \varkappa \text{ when } \xi \to \infty,$$

$$\Phi = \Phi_0(\xi), \quad \bar{h} = \bar{h}_0(\xi) \text{ when } f_1 = 0, \tag{19}$$

where $\Phi_0(\xi)$ and $\bar{h}_0(\xi)$ may be taken from the solution of (17) in the first approximation.

The solutions of system (19), being functions of the two arguments ξ and f_1, will depend on a number of constant parameters M_∞, $\bar{h}_W$, Pr, which express the conditions of specific problems. For each assigned value of these parameters the solution must be obtained in the whole range of variation of the parameter $\varkappa (0 \leq \varkappa < 1)$. It should be remembered here that $\varkappa$ is finally a known function of s, depending on the velocity distribution at the outer edge of the boundary layer.

The system of equations (19), written in finite differences, was solved on an electronic computer with the following values of the parameters:

$$M_\infty = 19.4, \quad \bar{h}_w = 0.0152, \quad \Pr = 0.712 \text{ and } 0 \leq \varkappa < 1.$$

In the numerical integration we chose a constant step for the variable $\xi : \Delta\xi = 0.05$, and a variable step for $f_1 : (\Delta f_1)_{max} = 5 \cdot 10^{-5}$, $(\Delta f_1)_{min} = 0.3125 \cdot 10^{-5}$.

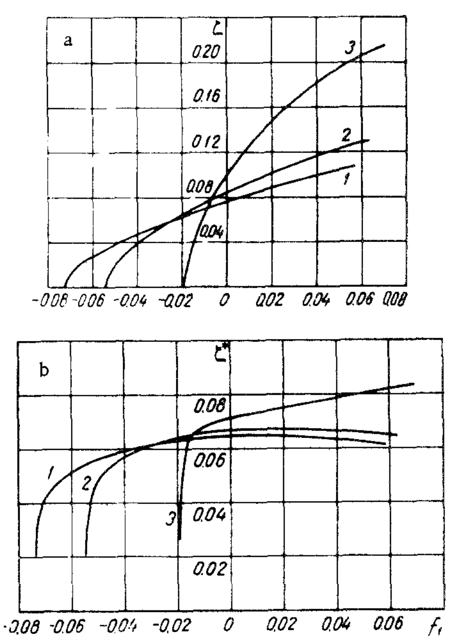

Fig. 3. Dependence of (a) ζ and (b) ζ^* on the shape factor f_1 for values of parameter $\varkappa$. 1) 0; 2) 0.5; 3) 0.9.

The calculation in the direction of ξ was done from the point $f_1 = 0$ to the point of separation of the boundary layer, where $\zeta = B\left(\frac{\partial^2 \Phi}{\partial \xi^2}\right)_{\xi=0} = 0$ (in the diffuser region, $f_1 < 0$), up to the forward stagnation point, where $F = 0$ (in the nozzle region, $f_1 > 0$).

The calculations carried out showed that when $f_1 \neq \neq 0$ all the characteristics of the boundary layer depend significantly on the local compressibility factor $\varkappa$.

It may be seen from Figs. 3a, b that when $\varkappa$ increases, the point of boundary layer separation moves upstream. At large values of $\varkappa$ there is even a change in the general nature of the dependence of the reduced heat flux to the wall ζ^* on the shape factor f_1.

On the basis of the solution obtained we may determine values of the derivatives $\partial\Phi/\partial\varkappa$, $\partial\bar{h}/\partial\varkappa$ and use them to obtain a solution to (17) in the second approximation.

NOTATION

x, y — longitudinal and transverse coordinates; u, v, — longitudinal and transverse velocities in the boundary layer; $Pr = \mu c_p/\lambda$ is the Prandtl number; μ is the dynamic viscosity; λ is the thermal conductivity; c_p is the specific heat of mixture at constant pressure; p is the pressure; ρ is the density; ν is the kinematic viscosity; $Le = \rho D C_p/\lambda$ is the Lewis number, D is the binary diffusion coefficient; h is the enthalpy; $\bar{h} = h/h_1$ is the dimensionless enthalpy; h_1 is the stagnation enthalpy in external flow; h_A, h_M is the enthalpy of atoms and molecules, respectively; g is the dimensionless total enthalpy; c_A is the concentration of atoms; ψ is the stream function; Φ is the 'reduced' stream function; M is the Mach number; f_k is the shape factor; x is the local compressibility factor. Subscripts: w is to denote conditions at the wall, e is at the outer edge of the boundary layer, 1 is in the adiabatically and isentropically decelerated external flow.

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